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RADIAL LIMITS OF n-SUBHARMONIC FUNCTIONS IN THE POLYDISC

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ABSTRACT. We prove a relation between a certain weighted radial limit of an n-subharmonic function in the polydisc U^n and the representing measure of its least n-harmonic majorant. We apply this result to functions in $N(U^n)$, the Nevalinna class of U^n. In particular, we obtain a necessary condition for a function to belong to the component of the origin in $N(U^n)$. These results are extensions of the work of J. H. Shapiro and A. L. Shields to $n > 1$.

1. Introduction. The results of the paper were motivated by the paper of J. H. Shapiro and A. L. Shields [3] concerning functions in the Nevanlinna class N. The space N consists of all function f analytic in the open unit disc satisfying $\sup_{0<r<1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta < \infty$. By the Canonical Factorization Theorem, every function $f \in N$ has a unique factorization

$$f = B(S_{\mu_1}/S_{\mu_2})F \tag{1.1}$$

where $B(z)$ is the Blaschke product with respect to the zeroes of $f(z)$, $F(z)$ is an outer function, and $S_{\mu_j}(z)$ are singular inner functions defined by

$$S_{\mu_j}(z) = \exp\left[-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu_j(t)\right]$$

where the μ_j are finite positive singular Borel measures on $[0, 2\pi)$.

In their paper [3], the authors proved that if $f \in N$, then for all t, $|t| = 1$,

$$\overline{\lim_{r\to 1^-}}\,(1-r)\log|f(rt)| = 2\sigma_f(\{t\}) \tag{1.2}$$

where $\sigma_f = \mu_2 - \mu_1$. The key in proving this result was in showing that if B is a Blaschke product, then

$$\overline{\lim_{r\to 1^-}}\,(1-r)\log|B(rt)| = 0, \qquad |t| = 1. \tag{1.3}$$

These results are then used to obtain information about topological properties of the space N.

In this paper we consider analogues of the above for functions in the Nevanlinna class in the polydisc U^n in $\mathbf{C}^n$. Unfortunately, there is no analogue of the factorization (1.1) for functions of several variables. However, as is shown in this paper, by

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using the least n-harmonic majorant of $\log|f(z)|$ one can obtain generalizations of (1.2) and many of the other results in [3] to functions in the Nevanlinna class in the polydisc. These results, however, are the consequence of more general results concerning n-subharmonic functions in the polydisc.

The key result of this paper, given as a lemma in §3, is as follows: if $F \not\equiv -\infty$ is n-subharmonic in U^n with $F(z) \leqslant 0$ and $\lim_{r\to 1} \int_{T^n} F(rt)\, d\lambda_n(t) = 0$, then

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j) \right) F(r_1 t_1, \ldots, r_n t_n) = 0$$

for all t on the distinguished boundary T^n. For the case $n = 1$, this gives (1.3) since $\log|B(z)|$ satisfies the hypothesis. The above result is then used to prove that if $f \not\equiv -\infty$ is n-subharmonic in U^n with $\sup_{0<r<1} \int_{T^n} f^+(rt)\, d\lambda_n(t) < \infty$, then

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j) \right) f(r_1 t_1, \ldots, r_n t_n) = 2^n \sigma_f(\{t\})$$

where σ_f is the singular part of the representing measure of f.

In §4, the results of §3 are applied to obtain generalizations of the results in [3] to functions in the Nevanlinna class $N(U^n)$. In particular, we obtain a necessary condition that a function f in $N(U^n)$ lie in the component of the origin in $N(U^n)$. §2 of this paper contains the necessary background material and notation.

2. Notation and preliminary results. Let U denote the unit disc in $\mathbf{C}$ with boundary T. For $n > 1$, U^n denotes the unit polydisc in $\mathbf{C}^n$ with distinguished boundary T^n, that is, $T^n = \{\zeta \in \mathbf{C}^n : |\zeta_j| = 1,\ 1 \leqslant j \leqslant n\}$. Also, we denote by λ the normalized Lebesgue measure on T and by λ_n the normalized Lebesgue measure on T^n.

A continuous real-valued function f on an open set in $\mathbf{C}^n$ is *n-harmonic* if f is harmonic in each variable z_j separately, $1 \leqslant j \leqslant n$. Also, a function f defined on an open set Ω in $\mathbf{C}^n$ with $-\infty \leqslant f < \infty$ is *n-subharmonic* if f is upper semicontinuous in Ω and subharmonic in each variable separately.

For $z \in U$, $t \in T$, the Poisson kernel on $U \times T$ is given by

$$P(z, t) = \frac{1 - |z|^2}{|t - z|^2}.$$

Also, for $z \in U^n$, $t \in T^n$, the *Poisson kernel* on $U^n \times T^n$ is given by

$$P(z, t) = \prod_{j=1}^{n} P(z_j, t_j).$$

If μ is a finite (signed) Borel measure on T^n, its *Poisson integral* is the function

$$P_z[d\mu] = \int_{T^n} P(z, t)\, d\mu(t).$$

The function $U(z) = P_z[d\mu]$ is n-harmonic in U^n.

For any function f in U^n, $f_r(t) = f(rt)$ if $0 \leqslant r < 1$ and $t \in T^n$; $f^*(t) = \lim_{r\to 1} f_r(t)$ for those $t \in T^n$ at which the limit exists; for real f, $f^+ = \max\{f, 0\}$.

The following lemma will be needed in the sequel.

LEMMA 2.1. *Suppose* $f \not\equiv -\infty$ *is n-subharmonic in* U^n *with* $\sup_{0<r<1} \int_{T^n} f_r^+ \, d\lambda_n < \infty$. *Then there exists a finite Borel measure* μ_f *on* T^n *such that*

$$(2.1) \qquad f(z) \leq \int_{T^n} P(z, t) \, d\mu_f(t)$$

and μ_f *is minimal among the Borel measures on* T^n *satisfying* (2.1). *Furthermore*,

$$(2.2) \qquad \lim_{r \to 1} \int_{T^n} f(rt)\phi(t) \, d\lambda_n(t) = \int_{T^n} \phi(t) \, d\mu_f(t)$$

for all ϕ *continuous on* T^n. *If*

$$(2.3) \qquad d\mu_f = h \, d\lambda_n + d\sigma_f$$

is the Lebesgue decomposition of μ_f *with* $h \in L^1(T^n)$ *and* σ_f *is singular*, *then* $h(t) = \lim_{r \to 1} P_{rt}[d\mu_f]$ *a.e. on* T^n. *Finally, if the family* $\{f_r^+ : 0 < r < 1\}$ *is uniformly integrable on* T^n, *then* $\sigma_f \leq 0$ *and*

$$(2.4) \qquad f(z) \leq P_z[h \, d\lambda_n].$$

The proof of the lemma may be found in [**2**]. An analogous result for plurisubharmonic functions on more general domains also holds [**4**].

The function $P_z[d\mu_f]$ in (2.1) is the *least n-harmonic majorant* of f. The measure μ_f will be referred to as the *representing measure* of the least n-harmonic majorant of f. From the above, if f satisfies the hypothesis of the lemma, then

$$(2.5) \qquad f^+(z) \leq P_z\left[h^+ \, d\lambda_n + d\sigma_f^+\right].$$

We will also need the following version of the maximum principle for n-subharmonic functions.

LEMMA 2.2. *Suppose* D_j, $1 \leq j \leq n$, *are bounded open connected subsets of* $\mathbf{C}$ *with boundary* B_j. *Let* $D = \times_{j=1}^n D_j$ *and* $B = \times_{j=1}^n B_j$. *If* f *is n-subharmonic in* D *and satisfies* $\limsup_{z \to \zeta} f(z) \leq 0$ *for every* $\zeta \in B$, *then* $f(z) \leq 0$ *for all* $z \in D$.

The proof is an immediate consequence of the maximum principle for subharmonic functions and hence is omitted.

3. Radial limits of n**-subharmonic functions.** In this section we investigate radial limits of n-subharmonic functions. The first result is a generalization of (1.3) to n-subharmonic functions in U^n.

LEMMA 3.1. *If* $F(z)$ *is n-subharmonic in* U^n ($F \not\equiv -\infty$) *satisfying*

(i) $F(z) \leq 0$ *for all* $z \in U^n$, *and*

(ii) $\lim_{r \to 1} \int_{T^n} F(rt) \, d\lambda_n(t) = 0$,

then for all $t \in T^n$,

$$\overline{\lim_{(r) \to (1)}} \left(\prod_{j=1}^n (1 - r_j) \right) F(r_1 t_1, \ldots, r_n t_n) = 0,$$

where $(r) = (r_1, \ldots, r_n)$, $0 < r_j < 1$.

PROOF. Without loss of generality we may assume that $t_j = 1$ for all $j = 1,\ldots,n$. Also, for convenience we give the proof only for $n = 2$ and indicate the modifications which need to be made for arbitrary n. Suppose that

$$\overline{\lim_{(r_1,r_2)\to(1,1)}}\ (1-r_1)(1-r_2)F(r_1,r_2) < 0.$$

Then there exist constants $A > 0$ and R, $0 < R < 1$, such that

$$F(r_1, r_2) \leqslant -AP(r_1, 1)P(r_2, 1) \tag{3.1}$$

for all $r_1, r_2 \geqslant R$. Let C_R denote the semicircle in the upper half of the unit disc passing through R and 1 with diameter $1 - R$. By considering the image of this semicircle under the conformal map $w = (1 + z)/(1 - z)$ it is clear that $P(\zeta, 1) = (1 + R)/(1 - R)$ for all $\zeta \in C_R$, $\zeta \neq 1$. Finally let D_R denote the open subset of U bounded by the semicircle C_R and the line segment from R to 1.

We will show that for all $(z_1, z_2) \in D_R \times D_R$

$$F(z_1, z_2) \leqslant -AP(z_1, 1)P(z_2, 1) + ABP(z_1, 1) + ABP(z_2, 1) + AB^2 \tag{3.2}$$

where $B = (1 + R)/(1 - R)$.

Before proving inequality (3.2), we show how this is used to obtain that

$$\lim_{r\to 1} \int_{T^n} F(rt)\, d\lambda_n(t) \leqslant -Ab$$

for some positive constant b, which is a contradiction. Fix a Stolz region S in U with vertex at 1 and let S^+ denote the intersection of S with the upper half of the disc. Then for r sufficiently close to 1, $S^+ \cap \{z \mid |z| = r\} \subset D_R$. Finally, let

$$I_r = \{e^{i\theta} \mid re^{i\theta} \in S^+\} \tag{3.3}$$

Then (3.2) gives

$$\int_{T^2} F(rt)\, d\lambda_2(t) \leqslant \int_{I_r^2} F(rt)\, d\lambda_2(t) \leqslant -A\left[\frac{1}{2\pi}\int_{I_r} P(re^{i\theta}, 1)\, d\theta\right]^2$$

$$+2\,AB|I_r|\left[\frac{1}{2\pi}\int_{I_r} P(re^{i\theta}, 1)\, d\theta\right] + AB^2|I_r|^2,$$

where $|I_r|$ denotes the length of I_r. A routine computation shows that $|I_r| \sim (1 - r)\tan(\beta/2)$ and

$$\lim_{r\to 1} \frac{1}{2\pi}\int_{I_r} P(re^{i\theta}, 1)\, d\theta = \frac{\beta}{2\pi}$$

where β is the angle of the Stolz domain at the vertex 1. Hence

$$\lim_{r\to 1} \int_{T^2} F(rt)\, d\lambda_2(t) \leqslant -A\left(\frac{\beta}{2\pi}\right)^2.$$

To complete the proof of the lemma we need to establish inequality (3.2). For $\varepsilon > 0$, consider the 2-subharmonic function H_ε given by

(3.4) $$H_\varepsilon(z_1, z_2) = F(z_1, z_2) + AP(z_1, 1)P(z_2, 1) - \varepsilon A w(z_1)P(z_2, 1) - \varepsilon AP(z_1, 1)w(z_2) - ABP(z_1, 1) - ABP(z_2, 1) - AB^2,$$

where for $z = re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, and fixed δ, $0 < \delta < 1$,

$$w(z) = \mathrm{Re}\left[e^{-i\pi/4}\frac{1+z}{1-z}\right]^{1+\delta} = \left|\frac{1+z}{1+z}\right|^{1+\delta}\cos(1+\delta)\left(\phi(z) - \frac{\pi}{4}\right).$$

Here $\phi(z) = \arg((1+z)/(1-z))$. For z in the upper half of the unit disc, $0 \leqslant \phi(z) \leqslant \pi/2$. Thus

(3.5) $$w(z) \geqslant \left|\frac{1+z}{1-z}\right|^{1+\delta}\rho, \qquad z \in D_R,$$

where $\rho = \cos(1+\delta)(\pi/4)$ which is positive. We now show that for all $\zeta = (\zeta_1, \zeta_2) \in \partial D_R \times \partial D_R$

(3.6) $$\overline{\lim_{z\to\zeta}}\, H_\varepsilon(z) \leqslant 0.$$

As the boundary of D_R consists of the semicircle C_R, the line segment $\Gamma_R = [R, 1)$ and the point $\{1\}$, there are nine cases to be considered, which by symmetry can be reduced to four cases. Note, $H_\varepsilon(\zeta)$ is defined on $\partial D_R \times \partial D_R$ for all ζ except when $\zeta_j = 1$ for some j.

(i) $(\zeta_1, \zeta_2) \in \Gamma_{\mathrm{R}} \times \Gamma_{\mathrm{R}}$.

Then by (3.1), $F(\zeta_1, \zeta_2) + AP(\zeta_1, 1)P(\zeta_2, 1) \leqslant 0$. Since all the remaining terms are negative, $H_\varepsilon(\zeta_1, \zeta_2) \leqslant 0$.

(ii) $(\zeta_1, \zeta_2) \in C_R \times C_R$, $\zeta_j \neq 1$, $j = 1, 2$.

Then since $P(\zeta_j, 1) = (1+R)/(1-R) = B$,

$$H_\varepsilon(\zeta_1, \zeta_2) \leqslant AP(\zeta_1, 1)P(\zeta_2, 1) - AB^2 = 0.$$

(iii) $(\zeta_1, \zeta_2) \in C_R \times \Gamma_R$, $\zeta_1 \neq 1$.

Then

$$H_\varepsilon(\zeta_1, \zeta_2) \leqslant AP(\zeta_1, 1)P(\zeta_2, 1) - ABP(\zeta_2, 1) = 0.$$

Similarly, if $(\zeta_1, \zeta_2) \in \Gamma_R \times C_R$, $\zeta_2 \neq 1$, $H_\varepsilon(\zeta_1, \zeta_2) \leqslant 0$.

(iv) $\zeta_1 = 1$, $\zeta_2 \in C_R \cup \Gamma_R$

(we also permit the possibility that $\zeta_2 = 1$). Then for $z \in D_R \times D_R$

$$H_\varepsilon(z) \leqslant AP(z_2, 1)[P(z_1, 1) - \varepsilon w(z_1)].$$

But for $z_1 \in D_R$, by (3.5),

$$P(z_1, 1) - \varepsilon w(z_1) \leqslant \left|\frac{1+z_1}{1-z_1}\right|\left(1 - \varepsilon\rho\left|\frac{1+z_1}{1-z_1}\right|^{\delta}\right),$$

and as $z_1 \to 1$, the term on the right becomes negative. Thus

$$\overline{\lim_{(z_1,z_2)\to(1,\zeta_2)}}\, H_\varepsilon(z_1, z_2) \leqslant 0,$$

and this holds for any $\zeta_2 \in D_R \cup \Gamma_R$. A similar argument holds for $\zeta_2 = 1$, $\zeta_1 \in C_R \cup \Gamma_R$. Therefore (3.6) holds for all $\zeta \in \partial D_R \times \partial D_R$ and hence by the maximum principle for n-subharmonic functions (Lemma 2.2) $H_\varepsilon(z) \leq 0$ for all $z \in D_R \times D_R$. Letting $\varepsilon \to 0$ we obtain inequality (3.2).

For arbitrary n, a similar argument will show that for all $z \in D_R^n$,

$$F(z) \leq -A \prod_{j=1}^{n} P(z_j, 1) + A \sum_{S \subsetneq \{1,\ldots,n\}} B^{n-|S|} \prod_{j \in S} P(z_j, 1). \tag{3.7}$$

As above, (3.7) is proved by considering the functions $H_\varepsilon(z)$, $\varepsilon > 0$, defined by

$$H_\varepsilon(z) = F(z) + A \prod_{j=1}^{n} P(z_j, 1) - \varepsilon A \prod_{j=1}^{n} w(z_j) \prod_{k \neq j} P(z_k, 1)$$
$$-A \sum_{S \subsetneq \{1,\ldots,n\}} B^{n-|S|} \prod_{j \in S} P(z_j, 1).$$

The remainder of the proof follows as above.

We now state and prove the main result of the paper.

THEOREM 3.2. *Let $f(z) \not\equiv -\infty$ be n-subharmonic in U^n satisfying*

$$\sup_{0<r<1} \int_{T^n} f^+(rt)\, d\lambda_n(t) < \infty.$$

Then for all $t \in T^n$,

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j) \right) f(r_1 t_1, \ldots, r_n t_n) = 2^n \sigma_f(\{t\}),$$

where as in Lemma 2.1, σ_f is the singular part of the representing measure μ_f of the least n-harmonic majorant of f.

PROOF. Let $H(z) = P_z[d\mu]$, where $d\mu = h d\lambda_n + d\sigma$ with $h \in L^1(T^n)$. We first show that

$$\lim_{(r)\to(1)} \left(\prod_{j=1}^{n} (1 - r_j) \right) H(r_1 t_1, \ldots, r_n t_n) = 2^n \sigma(\{t\}) \tag{3.8}$$

for all $t \in T^n$. By linearity it suffices to prove the result for μ nonnegative.

Fix $t \in T^n$. We first assume that $\sigma(\{t\}) = 0$. Let $\varepsilon > 0$ be given. Then there exist open intervals I_j, $j = 1, \ldots, n$, on the unit circle with $t_j \in I_j$ such that

$$\sigma(I) < \varepsilon \quad \text{and} \quad \int_I h\, d\lambda_n < \varepsilon$$

where $I = \times_{j=1}^{n} I_j$. Then

$$H(r_1 t_1, \ldots, r_n t_n) = \int_I \prod_{j=1}^{n} P(r_j t_j, \zeta_j) [h\, d\lambda_n(\zeta) + d\sigma(\zeta)]$$
$$+ \int_{T^n - I} \prod_{j=1}^{n} P(r_j t_j, \zeta_j)\, d\mu(\zeta)$$
$$\leq \prod_{j=1}^{n} \left(\frac{1 + r_j}{1 - r_j} \right) 2\varepsilon + \int_{T^n - I} \prod_{j=1}^{n} P(r_j t_j, \zeta_j)\, d\mu(\zeta).$$

On $T^n - I$, $\prod_{j=1}^n (1 - r_j) P(r_j t_j, \zeta_j) \leq 2^n$. Also if $\zeta \in T^n - I$, there exists k such that $\zeta_k \in T - I_k$ and thus

$$\lim_{(r)\to(1)} \prod_{j=1}^{n} (1 - r_j) P(r_j t_j, \zeta_j) = 0.$$

Hence

$$\lim_{(r)\to(1)} \left(\prod_{j=1}^{n} (1 - r_j) \right) H(r_1 t_1, \ldots, r_n t_n) \leq 2^{n+1}\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\lim_{(r)\to(1)} \left(\prod_{j=1}^{n} (1 - r_j) \right) H(r_1 t_1, \ldots, r_n t_n) = 0. \tag{3.9}$$

If $\sigma(\{t\}) = a > 0$, we consider the measure $\nu = \mu - a\delta_t$, where δ_t is point mass at t. Letting $V(z) = P_z[d\nu]$ we have

$$H(z) = V(z) + aP(z, t).$$

Since $\nu \geq 0$ and $\nu(\{t\}) = 0$, by (3.9),

$$\lim_{(r)\to(1)} \left(\prod_{j=1}^{n} (1 - r_j) \right) V(r_1 t_1, \ldots, r_n t_n) = 0.$$

Thus since $\lim_{(r)\to(1)} \prod_{j=1}^n (1 - r_j) P(rt_j, t_j) = 2^n$, (3.8) follows.

Let $F(z) = f(z) - H(z)$. Then F is n-subharmonic, less than or equal to zero, and by (2.2) satisfies $\lim_{r\to 1} \int_{T^n} F(rt)\, d\lambda_n(t) = 0$. Therefore by Lemma 3.1 and (3.8) we obtain that

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j) \right) f(r_1 t_1, \ldots, r_n t_n) = 2^n \sigma(\{t\}),$$

thus proving the result.

An immediate consequence of Theorem 3.2 is the following.

COROLLARY 1. *If $f \not\equiv -\infty$ is n-subharmonic in U^n with* $\sup_{0<r<1} \int_{T^n} f^+(rt)\, d\lambda_n(t) < \infty$, *then the representing measure of the least harmonic majorant of f is continuous at every $t \in T^n$ if and only if*

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j) \right) f(r_1 t_1, \ldots, r_n t_n) = 0$$

for all $t \in T^n$.

Recall that a measure μ is *continuous* at $t \in T^n$ if $\mu(\{t\}) = 0$.

For f n-subharmonic on T^n, $0 < r_j < 1$, let

$$M_\infty(f; (r)) = M_\infty(f; r_1, \ldots, r_n) = \sup_{t \in T^n} f(r_1 t_1, \ldots, r_n t_n). \tag{3.10}$$

Then the following holds.

COROLLARY 2. *Suppose* $f \not\equiv -\infty$ *is n-subharmonic with*

$$\sup_{0<r<1} \int_{T^n} f^+(rt)\, d\lambda_n(t) < \infty \quad \text{and} \quad \overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1-r_j) \right) f(r_1t_1,\ldots,r_nt_n) = 0$$

for all $t \in T^n$. *Then*

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1-r_j) \right) M_\infty(f; r_1,\ldots,r_n) = 0. \tag{3.11}$$

PROOF. Let A denote the limit superior in (3.11). Then there exist sequences $\{r_{j(k)}\}_{k=1}^{\infty}, j = 1,\ldots,n$, such that

$$\left(\prod_{j=1}^{n} (1-r_{j(k)}) \right) M_\infty(f; (r_{j(k)})) \to A.$$

Let $H(z) = P_z[d\mu]$ be the least n-harmonic majorant of f, where μ by hypothesis is continuous on T^n. By the maximum principle, for each $(r_{j(k)})$, there exists a point $(t_{j(k)}) \in T^n$ such that

$$\begin{aligned} M_\infty(f; (r_{j(k)})) &= f(r_{1(k)}t_{1(k)},\ldots,r_{n(k)}t_{n(k)}) \\ &\leqslant \int_{T^n} \prod_{j=1}^{n} P(r_{j(k)}t_{j(k)}, \zeta_j)\, d\mu(\zeta). \end{aligned}$$

Since T^n is compact, by choosing subsequences if necessary, we may assume that $(t_{j(k)})$ converges to some $t \in T^n$. Since μ is continuous at t, as in the proof of Theorem 3.2,

$$\lim_{k\to\infty} \left(\prod_{j=1}^{n} (1-r_{j(k)}) \right) \int_{T^n} \prod_{j=1}^{n} P(r_{j(k)}t_{j(k)}, \zeta_j)\, d\mu(\zeta) = 0.$$

Thus

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1-r_j) \right) M_\infty(f; r_1,\ldots,r_n) \leqslant 0.$$

However, since $f(r_1t_1,\ldots,r_nt_n) \leqslant M_\infty(f; (r))$, we obtain equality.

For a Borel measure μ on T^n, we denote by μ_j the projection of the measure μ onto the jth coordinate T, that is, if $E \subset T$ is a Borel set,

$$\mu_j(E) = \mu\left(\mathop{\times}_{k=1}^{n} T_k \right) \tag{3.12}$$

where $T_k = T$ for all $k \neq j$ and $T_j = E$. Also, if f is n-subharmonic, $0 < r < 1$, we define $f_r^j(w)$, $w \in U$, by

$$f_r^j(w) = \int_{T^{n-1}} f(\underbrace{rt_1,\ldots,w,\ldots,rt_n}_{j\text{th coordinate}})\, d\lambda_{n-1}(t). \tag{3.13}$$

Then for each $j = 1,\ldots,n$, and each r, $0 < r < 1$, f_r^j is subharmonic in U. Furthermore, if $0 < r_1 < r_2 < 1$, then

$$f_{r_1}^j(w) \leqslant f_{r_2}^j(w), \qquad w \in U.$$

This follows from the fact that for fixed z_j, the function $\zeta \to f(\zeta_1,\dots,z_j,\dots,\zeta_n)$ is $(n-1)$-subharmonic in U^{n-1}.

THEOREM 3.3. *Let $f \not\equiv -\infty$ be n-subharmonic in U^n satisfying*

$$\sup_{0<r<1} \int_{T^n} f^+(rt)\, d\lambda_n(t) < \infty,$$

and let μ be the representing measure of the least n-harmonic majorant of f. Then

$$\varlimsup_{r\to 1} (1-r) f_r^j(rt) = 2\mu_j(\{t\}), \qquad t \in T. \tag{3.14}$$

PROOF. Let $H(z) = P_z[d\mu]$, $z \in U^n$. Also, for $w \in U$, $0 < r < 1$, set $F_r(w) = f_r^j(w) - H^j(w)$ where

$$H^j(w) = H(\underbrace{0,\dots,w,\dots,0}_{j\text{th coordinate}}).$$

Then $F_r(w)$ is subharmonic in U, $F_r(w) \leqslant 0$, and

$$\int_T F_r(rt_j)\, d\lambda(t_j) = \int_T f_r^j(rt_j)\, d\lambda(t_j) - H(0)$$
$$= \int_{T^n} f(rt)\, d\lambda_n(t) - \mu(T^n).$$

Thus by Lemma 2.1, $\lim_{r\to 1} \int_T F_r(rt_j)\, d\lambda(t_j) = 0$.

We shall prove the following two statements.

(i) $\lim_{r\to 1}(1-r)H^j(rt_j) = 2\mu_j(\{t_j\})$, and
(ii) $\varlimsup_{r\to 1}(1-r)F_r(rt_j) = 0$.

By definition of μ_j, $\int_T \phi(t_j)\, d\mu_j(t_j) = \int_{T^n} \phi(t_j)\, d\mu(t)$, for any ϕ continuous on T. Thus

$$H^j(w) = H(0,\dots,w,\dots,0) = \int_{T^n} P(w,t_j)\, d\mu(t) = \int_T P(w,t_j)\, d\mu_j(t_j).$$

Hence by (3.8), for the case $n = 1$, $\lim_{r\to 1}(1-r)H^j(rt_j) = 2\mu_j(\{t_j\})$, thus proving (i).

Suppose that $\varlimsup_{r\to 1}(1-r)F_r(rt_j) < 0$. Again, without loss of generality we may assume $t_j = 1$. Then there exist constants $A > 0$ and r_0, $0 < r_0 < 1$, such that

$$F_r(r) < -AP(r,1), \qquad r \geqslant r_0.$$

Let $\varepsilon > 0$ be given. Since $\int_T F_r(rt)\, d\lambda(t) \to 0$, we can choose $R \geqslant r_0$ such that

$$-\varepsilon < \int_T F_r(rt)\, d\lambda(t)$$

for all $r \geqslant R$. Since $F_R \leqslant F_r$ where $r \geqslant R$, we have

$$F_R(r) \leqslant -AP(r,1) \quad \text{for } r \geqslant R.$$

As in the proof of Lemma 3.1,

$$F_R(z) < -AP(z,1) + AP(R,1)$$

for all $z \in D_R$. Now for $r > R$,

$$-\varepsilon < \int_T F_R(Rt)\,d\lambda(t) \leqslant \int_T F_R(rt)\,d\lambda(t) \leqslant \int_{I_r} F_R(rt)\,d\lambda(t)$$

$$< -A\int_{I_r} P(rt,1)\,d\lambda(t) + AP(R,1)\lambda(I_r),$$

where I_r is defined as in (3.3). Letting $r \to 1$ we obtain $-\varepsilon < -A\beta/2\pi$ where β is the angle of the Stolz domain at the vertex. But ε can be chosen arbitrarily small. This contradiction proves (ii). Thus

$$\overline{\lim_{r\to 1}}\,(1-r)f_r^j(rt) = 2\mu_j(\{t\}) \quad \text{for all } t \in T.$$

4. Functions of bounded characteristic. In this section we apply the results of the previous section to obtain some results on functions of bounded characteristic in U^n. As in the case for one variable, the Nevanlinna class $N(U^n)$ is the algebra of functions f holomorphic in U^n for which

$$\sup_{0<r<1} \int_{T^n} \log^+|f(rt)|\,d\lambda_n(t) < \infty.$$

Also, we consider the subspace $N_*(U^n)$ consisting of all functions $f \in N(U^n)$ for which the family $\{\log^+|f_r| : 0 < r < 1\}$ is uniformly integrable on T^n. By Lemma 2.1, if $f \in N(U^n)$, then there exists a minimal Borel measure μ on T^n such that

(4.1) $$\log|f(z)| \leqslant P_z[d\mu]$$

with $d\mu = \log|f^*|\,d\lambda_n + d\sigma$, where σ is singular with respect to λ_n and $f^*(t) = \lim_{r\to 1} f(rt)$ a.e. on T^n. Functions in $N_*(U^n)$ are characterized by the condition that $\sigma \leqslant 0$ and that

$$\lim_{r\to 1} \int_{T^n} \log^+|f(rt)|\,d\lambda_n(t) = \int_{T^n} \log^+|f^*|\,d\lambda_n.$$

For $f \in N(U^n)$, we define $\|f\|$ by

(4.2) $$\|f\| = \lim_{r\to 1} \int_{T^n} \log(1 + |f(rt)|)\,d\lambda_n(t).$$

As in the one variable case, by setting $d(f, g) = \|f - g\|$ one obtains a complete translation invariant metric on $N(U^n)$ which induces a topology stronger than that of uniform convergence on compact sets. For $f, g \in N_*(U^n)$,

$$d(f,g) = \int_{T^n} \log(1 + |f^* - g^*|)\,d\lambda_n$$

and (N^*, d) is a complete topological vector space in which multiplication is continuous **[5]**.

Let $f \in N(U^n)$. For convenience, we will refer to the representing measure μ of the least n-harmonic majorant of $\log|f|$ as the *boundary measure of* f. Also, if σ denotes the singular part of μ, we denote by σ^+ and σ^- the positive and negative variations of σ.

PROPOSITION 4.1. *Let $f \in N(U^n)$. Then*

$$\|f\| = \int_{T^n} \log(1 + |f^*|)\, d\lambda_n + \sigma^+(T^n) \tag{4.3}$$

and

$$\lim_{a\to 0} \|af\| = \sigma^+(T^n). \tag{4.4}$$

The proof of Proposition 4.1 is the same as the proofs of the corresponding results for the case $n = 1$ as given in [**3**], and hence is omitted.

An immediate consequence of (4.4) is that scalar multiplication is not continuous. Also, as was mentioned in [**3**], every finite-dimensional linear subspace of N/N_* has the discrete topology and, as a consequence, every finite-dimensional linear subspace of N which intersects N_* only at the origin also has the discrete topology.

One immediate consequence of Theorem 3.2 is the following

PROPOSITION 4.2. *Let $f \in N(U^n)$ and let σ_f be the singular part of the boundary measure of f. Then*

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j)\right) \log|f(r_1t_1,\ldots,r_nt_n)| = 2^n\sigma_f(\{t\}) \tag{4.5}$$

and

$$\overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j)\right) \log^+|f(r_1t_1,\ldots,r_nt_n)| = 2^n\sigma_f^+(\{t\}). \tag{4.6}$$

In [**3**], the authors defined the following subadditive, continuous, "nonarchimedian" functional λ_t on $N(U)$, $t \in T$, by

$$\lambda_t(f) = \overline{\lim_{r\to 1}}\, (1 - r)\log^+|f(rt)|.$$

By (4.6), $\lambda_t(f) = 2\sigma_f^+(\{t\})$. The functional λ_t was used to show that $N(U)$ was disconnected. In [**3**, Theorem 2.1], the authors showed that if $\lambda_t(f) \neq \lambda_t(g)$ for some t, then f and g lie in different components of $N(U)$. In [**1**], J. W. Roberts showed that the component of the origin in $N(U)$ consists of all functions f for which $\lambda_t(f) = 0$ for all $t \in T$. This corresponds to the set of all functions f in $N(U)$ for which σ^+ is continuous on T.

Suppose $f \in N(U^n)$. As in [**3**], for $t \in T^n$ we define $\lambda_t(f)$ by

$$\lambda_t(f) = \overline{\lim_{(r)\to(1)}} \left(\prod_{j=1}^{n} (1 - r_j)\right) \log^+|f(r_1t_1,\ldots,r_nt_n)|. \tag{4.7}$$

By (4.6), $\lambda_t(f) = 2^n\sigma^+(\{t\})$, where σ^+ is the positive variation of the singular part of the boundary measure of f. By (4.3) we have

$$\lambda_t(f) \leqslant 2^n\|f\| \tag{4.8}$$

for all $t \in T^n$. Thus λ_t is continuous.

From the inequalities,

$$\log^+(x+y) \leqslant \log^+ x + \log^+ y + \log 2 \qquad (x, y \geqslant 0),$$
$$\log^+(x+y) \leqslant \max\{\log^+ x, \log^+ y\} + \log 2,$$

it follows from (4.7) that λ_t is subadditive and

$$\lambda_t(f+g) \leqslant \max\{\lambda_t(f), \lambda_t(g)\}.$$

Hence if $\lambda_t(f) \leqslant \lambda_t(g)$,

$$\lambda_t(g) = \lambda_t(g+f-f) \leqslant \max\{\lambda_t(f+g), \lambda_t(f)\} \leqslant \lambda_t(g).$$

Therefore

$$\lambda_t(f+g) = \max\{\lambda_t(f), \lambda_t(g)\}. \tag{4.9}$$

This is referred to as the nonarchimedian property of λ. As in [**3**] we obtain the following

PROPOSITION 4.3. *If $f, g \in N(U^n)$ and $\lambda_t(f) \neq \lambda_t(g)$ for some $t \in T^n$, then f and g lie in different components of $N(U^n)$.*

PROOF. The proof is similar to the proof of Theorem 2.1 in [**3**]. For completeness we include a sketch of the proof. Suppose $\lambda_t(g) < a < \lambda_t(\mathrm{f})$. Since λ_t is continuous, the set

$$V = \{h \in N(U^n) : \lambda_t(h) > a\}$$

is open, contains f but not g. Furthermore, by the nonarchimedian property of λ_t and the inequality $2^n\|h\| \geqslant \lambda_t(h)$, it follows that for every $h_0 \in N(U^n) - V$,

$$\{h \in N(U^n) : \|h_0 - h\| < a/2^n\} \cap V = \varnothing.$$

Thus V is closed.

If we let $C_n(N)$ denote the component of the origin in $N(U^n)$, then as a consequence of Proposition 4.3 we obtain the following

COROLLARY. $C_n(N) \subset \{f \in N(U^n) \mid \lambda_t(f) = 0$ *for all* $t \in T^n\}$.

For the case $n = 1$, the result due to J. W. Roberts [**1**] shows that $f \in C_1(N)$ if and only if $\lambda_t(f) = 0$ for all $t \in T^n$. At this time we have been unable to obtain a characterization of $C_n(N)$ for $n > 1$ and we suspect that the analogue of the case $n = 1$ does not hold for $n > 1$.

We conclude this section with the following result concerning products of functions in $C_n(N)$.

PROPOSITION 4.4. *If $f \in C_n(N)$ and $g \in N_*(U^n)$, then $fg \in C_n(N)$.*

PROOF. Fix $g \in N_*(U^n)$ and consider the map $\phi_g\colon N(U^n) \to N(U^n)$ by $\phi_g(f)(z) = f(z)g(z)$. We will show that ϕ_g is continuous. Suppose $f_j \to f$ in $N(U^n)$. Then from (4.3) we obtain that both $\int_{T^n} \log(1 + |f_j^* - f^*|)\, d\lambda_n$ and $\sigma^+_{f_j - f}(T^n)$ converge to zero as $j \to \infty$. Here $\sigma^+_{f_j - f}$ denotes the positive variation of the singular part of the boundary measure of $f_j - f$. Consider $\|\phi_g(f_j) - \phi_g(\mathrm{f})\| = \|\phi_g(f_j - f)\|$. By (4.3),

$$\left\|\phi_g(f_j - f)\right\| = \int_{T^n} \log\left(1 + |g^*|\,|f_j^* - f^*|\right) d\lambda_n + \sigma^+_{g(f_j - f)}(T^n).$$

Since $\int_{T^n} \log(1 + |f_j^* - f^*|)\, d\lambda_n \to 0$ and $\log(1 + |g^*|)$ is integrable, it follows that

$$\lim_{j \to \infty} \int_{T^n} \log\left(1 + |g^*|\,|f_j^* - f^*|\right) d\lambda_n = 0.$$

Also, since $\log^+|g(f_j - f)| \leq \log^+|f_j - f| + \log^+|g|$ and since the representing measure of the least n-harmonic majorant of $\log^+|g(z)|$ is absolutely continuous, $\sigma^+_{g(f_j - f)} \leq \sigma^+_{f_j - f}$. Therefore $\sigma^+_{g(f_j - f)}(T^n) \to 0$ and thus ϕ_g is continuous. But then $\phi_g(C_n)$ is connected and contains zero. Hence $\phi_g(C_n) \subset C_n$, proving the result.

REMARK. In the case $n = 1$, if $f, g \in C_1(N)$, then $fg \in C_1(N)$. This however is a consequence of the characterization of $C_1(N)$ given in **[1]** and does not follow from any continuity property. Even for fixed $g \in C_1(N)$, $g \notin N_*$, the mapping $f \to fg$ is no longer continuous. One is tempted to conjecture that $C_n(N)$ is also closed under products if $n > 1$. Unfortunately we have been unable to prove this.

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